ISRAEL JOURNAL OF MATHEMATICS, Vol. 55, No. 3, 1986

A REMARK ON REGULARIZATION IN HILBERT SPACES

BY

J. M. LASRY AND P. L. LIONS

CEREMADE, Université Paris — Dauphine, Place de Lattre de Tassigny, 75775 Paris Cedex 16, France

ABSTRACT

We present here a simple method to approximate uniformly in Hilbert spaces uniformly continuous functions by $C^{1,1}$ functions. This method relies on explicit inf-sup-convolution formulas or equivalently on the solutions of Hamilton–Jacobi equations.

Introduction

Let H be a Hilbert space and let us denote by $|\cdot|$ and $(\cdot,\cdot)$ its norm and scalar product respectively. Let $u \in \mathrm{BUC}(H)$ — space of bounded uniformly continuous scalar functions. The problem we consider here concerns the approximation of u by a sequence u_ε of functions in $C_b^1(H)$ or even $C_b^{1,1}(H)$† such that u_ε converges uniformly on H to u. The usual way to find u_ε in the case when H is finite dimensional is to use convolution with smooth kernels: this method is not only explicit but enjoys a few important properties like, for example:

$$\sup_H |\nabla u_\varepsilon| \leqq C_\varepsilon \sup_H |u|, \tag{1}$$

$$\sup_{x \neq y} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)|\,|x-y|^{-1} \leqq C_\varepsilon \sup_H |u|, \tag{2}$$

$$\inf_H u \leqq u_\varepsilon \leqq \sup_H u \qquad \text{on } H, \tag{3}$$

$$\sup_H |\nabla u_\varepsilon| \leqq \sup_{x \neq y} |u(x) - u(y)|\,|x-y|^{-1} \leqq +\infty. \tag{4}$$

† $C_b^1(H) = \{v \in C^1(H),\ v,\ \nabla v \text{ bounded on } H\}$; $C_b^{1,1}(H) = \{v \in C_b^1(H),\ \nabla v \text{ Lipschitz on } H\}$.

Received July 17, 1985 and in revised form February 3, 1986

In addition, the regularization commutes with translations, is uniformly bounded in $C_b^{1,1}$ if $u \in C_b^{1,1}$ and it is order-preserving ... .

Unfortunately, this method breaks down when H is infinite dimensional. Our goal here is to present a simple method which works for arbitrary Hilbert spaces and which still enjoys properties (1)–(4), which commutes with translations, preserves order ... . We have in fact an explicit formula for the approximations u_ε: indeed, we prove in section I below that

$$\underline{u}_\varepsilon(x) = \sup_{z \in H} \inf_{y \in H} \left[u(y) + \frac{1}{2\varepsilon} |z - y|^2 - \frac{1}{\varepsilon} |z - x|^2 \right]$$

as well as

$$\bar{u}_\varepsilon(x) = \inf_{z \in H} \sup_{y \in H} \left[u(y) - \frac{1}{2\varepsilon} |z - y|^2 + \frac{1}{\varepsilon} |z - x|^2 \right]$$

are elements of $C_b^{1,1}$, that they satisfy (1)–(4) and in addition

(5) $$\underline{u}_\varepsilon \leqq u \leqq \bar{u}_\varepsilon \qquad \text{on } H$$

and $\bar{u}_\varepsilon$, $\underline{u}_\varepsilon$ converge uniformly on H to u.

There exist other approximation methods valid in infinite dimension, but we are not aware of any other method satisfying (1)–(4), or preserving translation and order, or as explicit as the inf-sup-convolution formula. In [7] A. S. Nemirovskii and S. M. Semenov have proved that functions of BUC(B) — where B is the unit ball of an Hilbert space H — can be uniformly approximated on B by functions in the class $C_b^{1,1}(B)$ (see (viii) below at the end of part I), and that there exist functions in BUC(B) which cannot be uniformly approximated by functions with uniformly continuous second order derivatives. Hence the space $C_b^{1,1}$ is the natural space for uniform approximation of BUC functions by more regular functions.

Let us mention that the main difference with convolution type regularizations (in finite dimensions) consists in the nonlinearity of the above method.

At this stage, we would like to make a few remarks on $\underline{u}_\varepsilon$, $\bar{u}_\varepsilon$ and in particular we wish to pinpoint the relations with Hamilton–Jacobi equations. Indeed, consider the following equations:

(6) $$\frac{\partial u}{\partial t} + \tfrac{1}{2} |\nabla u|^2 = 0 \quad \text{in } H \times]0, +\infty[, \qquad u\big|_{t=0} = v \quad \text{in } H,$$

resp. (7) $$\frac{\partial u}{\partial t} - \tfrac{1}{2} |\nabla u|^2 = 0 \quad \text{in } H \times]0, +\infty[, \qquad u\big|_{t=0} = v \quad \text{in } H;$$

where H is, to simplify, finite dimensional and $v \in \mathrm{BUC}(H)$. Observe that, formally, (7) is obtained from (6) by "reserving time". Then, it is known that the "right solutions" of (6) (resp. (7)), namely the viscosity solutions introduced by M. G. Crandall and P. L. Lions [2] — see also for further properties M. G. Crandall, L. C. Evans and P. L. Lions [4] — are given by the Lax–Oleinik formula:

$$u(x,t)=\inf_{y\in H}\left\{v(y)+\frac{1}{2t}|x-y|^2\right\}, \tag{8}$$

resp.

$$u(x,t)=\sup_{y\in H}\left\{v(y)-\frac{1}{2t}|x-y|^2\right\}, \tag{9}$$

and these solutions form a semigroup that we denote by $S_F(t)$ (resp. $S_{-F}(t)$) where $F(p)=\frac{1}{2}|p|^2$: for a proof of these facts we refer to P. L. Lions [6].

We observe next that the proposed regularized functions are nothing but:

$$\underline{u}_\varepsilon = S_{-F}\left(\frac{\varepsilon}{2}\right) S_F(\varepsilon)u, \qquad \bar{u}_\varepsilon = S_F\left(\frac{\varepsilon}{2}\right) S_{-F}(\varepsilon)u.$$

In fact, as we will see later on, we could as well introduce some two-parameters approximation of u, namely

$$\underline{u}_{\varepsilon,\delta} = S_{-F}(\delta)S_F(\varepsilon)u, \qquad \bar{u}_{\varepsilon,\delta} = S_F(\delta)S_{-F}(\varepsilon)u,$$

choosing $0<\delta<\varepsilon$.

Let us emphasize that (7) corresponds only formally to a time reversal of (6) and that in general (because shocks are forming and entropy increases) $S_{-F}(\delta)S_F(\varepsilon)u$ does not coincide with $S_F(\varepsilon-\delta)u$. The equality holds essentially in the case of smooth u, say u in $C^{1,1}(H)$, in which case we do have for ε small enough:

$$\underline{u}_{\varepsilon,\delta} = S_F(\varepsilon-\delta)u, \qquad \bar{u}_{\varepsilon,\delta} = S_{-F}(\varepsilon-\delta)u$$

and thus $\underline{u}_{\varepsilon,\delta},\ \bar{u}_{\varepsilon,\delta} \to u$ as $\delta \to \varepsilon$.

The reason for the regularity of $\underline{u}_\varepsilon$, $\bar{u}_\varepsilon$ (or $\bar{u}_{\varepsilon,\delta}, \underline{u}_{\varepsilon,\delta}$) is the following: if $v \in C_b(H)$ then $S_F(t)v$ (resp. $S_{-F}(t)v$) is for $t>0$ in $W^{1,\infty}(H)$ and semi-concave (resp. semi-convex) and more precisely we have

$$S_F(t)v - \frac{1}{2t}|x-x_0|^2 \quad \text{is concave for all } x_0 \in H$$

(resp. $S_{-F}v+(1/2t)|x-x_0|^2$ is convex for all $x_0 \in H$). Such results first considered in P. L. Lions [6] are elementary observations that we recall in section II below. Hence, $\underline{u}_{\varepsilon,\delta}$ (for instance) is for any $\delta > 0$ semi-convex but, in addition, since $S_F(\varepsilon)u$ is semi-concave for all $\varepsilon > 0$ with "second derivatives" bounded by $1/\varepsilon$ it is not difficult to check on the characteristics (at least formally) that for $\delta < \varepsilon$, $S_{-F}(\delta)[S_F(\varepsilon)u]$ is still semi-concave. And this yields the $C_b^{1,1}$ regularity! This second step has already been observed in I. Ekeland and J. M. Lasry [5]. Let us also mention that *if* v is convex, then $S_F(t)v$ is nothing else than the Yosida approximation of v (of order t) and it is well-known that $S_F(t)v \in C_b^{1,1}(H)$.

Note also that the kernel $\Phi(y) = |y|^2$ could be replaced in the inf-sup-formula by any convex, even, C^2 function (with $\Phi(0) = 0$) such that $\Phi''(x) \geqq c \,\mathrm{Id}$, $c > 0$, for all x in a neighborhood of 0.

We conclude this introduction by mentioning that our motivation for the regularization problem comes from the study of Hamilton–Jacobi equations in infinite dimensional spaces which is being developed by Barbu and Da Prato [1], M. G. Crandall and P. L. Lions [3] and that the above explicit regularization ideas are being applied in [3].

Let us finally mention that everywhere below we identify H with its dual.

I. Main properties of the regularizations

Let $u \in \mathrm{BUC}(H)$, i.e. assume there exists m continuous, nondecreasing on $[0,\infty[$ such that: $m(0) = 0$, $m(t+s) \leqq m(t) + m(s)$ for $s, t \geqq 0$ and:

$$|u(x) - u(y)| \leqq m(|x-y|), \qquad \text{for all } x, y \in H. \tag{10}$$

We consider for $0 < \delta < \varepsilon$, $x \in H$

$$\underline{u}_{\varepsilon,\delta} = S_{-F}(\delta)S_F(\varepsilon)u = \sup_{z\in H}\inf_{y\in H}\left[u(y) + \frac{1}{2\varepsilon}|z-y|^2 - \frac{1}{2\delta}|z-x|^2\right],$$

$$\bar{u}_{\varepsilon,\delta} = S_F(\delta)S_{-F}(\varepsilon)u = \inf_{z\in H}\sup_{y\in H}\left[u(y) - \frac{1}{2\delta}|z-y|^2 + \frac{1}{2\varepsilon}|z-x|^2\right].$$

THEOREM. *The functions $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ belong to $C^{1,1}(H)$. Let t_ε be the maximum positive root of: $t_\varepsilon^2 = 2\varepsilon\, m(t_\varepsilon)$, so that $t_\varepsilon \varepsilon^{-1/2} \to 0$ as $\varepsilon \to 0$. We have the following inequalities:*

$$-\infty \leqq \inf_H u \leqq \underline{u}_{\varepsilon,\delta} \leqq u \leqq \bar{u}_{\varepsilon,\delta} \leqq \sup_H u \leqq \infty \qquad \textit{on } H; \tag{11}$$

$$\sup_H |\underline{u}_{\varepsilon,\delta} - u| \leqq m(t_\varepsilon); \qquad \sup_H |\bar{u}_{\varepsilon,\delta} - u| \leqq m(t_\varepsilon); \tag{12}$$

(13) $|\underline{u}_{\varepsilon,\delta}(x)-\underline{u}_{\varepsilon,\delta}(y)|\leqq m(|x-y|), \qquad |\bar{u}_{\varepsilon,\delta}-u|\leqq m(t_\varepsilon+t_\delta)+\dfrac{t_\delta^2}{2\delta}$;

(14) $\sup_H |\nabla \underline{u}_{\varepsilon,\delta}|\leqq \dfrac{t_\varepsilon}{\varepsilon}$; $\quad \sup_H |\nabla \bar{u}_{\varepsilon,\delta}|\leqq \dfrac{t_\varepsilon}{\varepsilon}$;

(15) $|\nabla \underline{u}_{\varepsilon,\delta}(x)-\nabla \underline{u}_{\varepsilon,\delta}(y)|\leqq C_{\varepsilon,\delta}|x-y|, \qquad |\nabla \bar{u}_{\varepsilon,\delta}(x)-\nabla \bar{u}_{\varepsilon,\delta}(y)|\leqq C_{\varepsilon,\delta}|x-y|$

for all $x, y \in H$, *where* $C_{\varepsilon,\delta}=\mathrm{Max}(\delta^{-1},(\varepsilon-\delta)^{-1})$. ■

REMARKS. (i) If $u \in C^{1,1}(H)$, $\nabla u \in W^{1,\infty}(H)$, then $\underline{u}_{\varepsilon,\delta}=S_F(\varepsilon-\delta)u$ for ε small enough (while $\bar{u}_{\varepsilon,\delta}=S_{-F}(\varepsilon-\delta)u$) and $\nabla \underline{u}_{\varepsilon,\delta}$ remains uniformly bounded in $W^{1,\infty}(H)$ for ε small enough.

If $u \in \mathrm{BUC}(H)$ is Holder continuous then so are $\underline{u}_{\varepsilon,\delta}$ and $\bar{u}_{\varepsilon,\delta}$, and $|\underline{u}_{\varepsilon,\delta}|_\alpha \leqq |u|_\alpha$ and $|\bar{u}_{\varepsilon,\delta}|_\alpha \leqq |u|_\alpha$ with:

$$|v|_\alpha=\mathrm{Sup}\{|v(x)-v(y)|/|x-y|^\alpha ;\, x,y\in H, x\neq y\}$$

(this is a particular case of (13)). As a general rule if $u \in \mathrm{BUC}(H)$ enjoys more regularity, the functions $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ will (usually) also enjoy more regularity. For example, it can be shown that if $u \in C^{2+p}(H)$, $p \in \mathbf{N}$, $\underline{u}_{\varepsilon,\delta}$ and $\bar{u}_{\varepsilon,\delta}$ belong also to $C^{2+p}(H)$.

(ii) Clearly, the regularizations commute with translations and they preserve order (if $u \leqq v$ on H, then $\underline{u}_{\varepsilon,\delta}\leqq \underline{v}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}\leqq \bar{v}_{\varepsilon,\delta}$).

(iii) If $u \in C_b(H)$, then $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}\in C_b^{1,1}(H)$ and they converge to u pointwise in H as ε, $\delta\to 0$. More generally, if $u \in C(H)$ and satisfies

$$|u(x)|\leqq C(1+|x|^2) \qquad \text{on } H$$

then for ε small enough (and $0<\delta<\varepsilon$) $\bar{u}_{\varepsilon,\delta}$, $\underline{u}_{\varepsilon,\delta}\in C^{1,1}(H)$, they converge pointwise to u as $\varepsilon,\delta\to 0$, and $\nabla \bar{v}_{\varepsilon,\delta}$ may be bounded together with its Lipschitz modulus on balls by constants depending only on the growth of u on balls .... In addition if u is uniformly continuous on balls $\bar{B}_R$, one checks easily that $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ converge uniformly on balls to u. Finally if u is lower-semicontinuous and bounded below, then $\underline{u}_{\varepsilon,\delta}\in C^{1,1}(H)$ for $0<\delta<\varepsilon$, and $\underline{u}_{\varepsilon,\delta}$ converges pointwise to u when $\varepsilon,\delta\to 0$.

(iv) If one is only interested in regularizing functions in UC(H) into Lipschitz functions, it is enough to consider :

$$u_\varepsilon(x)=\inf_{y\in H}\left[u(y)+\frac{1}{\varepsilon}|x-y|^p\right]$$

for any $p\geqq 1$ (if $p=1$, one has to take ε small enough) — and one may replace $(1/\varepsilon)|x|^p$ by $(1/\varepsilon)\Phi(|x|)$ for a general Φ even, convex, $\Phi(0)=0$ and $\Phi\to+\infty$ as

$t \to +\infty$. In addition, let us mention that this regularization works in an arbitrary Banach space (or even metric spaces, take $(1/\varepsilon)d(x, y)$).

(v) Let us finally mention a few additional properties of the above regularization: first of all, if u is convex (resp. concave) then $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ are also convex (resp. concave). Indeed we just have to prove that if u is convex then $S_F(\varepsilon)u$ is convex. But observing that $u(y) + (1/2\varepsilon)|x - y|^2$ is jointly convex in (x, y), and using the lemma in section II, we see that $S_F(\varepsilon)u$ is convex. The second property we wish to mention concerns a subsolution of convex Hamilton–Jacobi equations: let $F \in C(H)$ be convex, let $f \in \mathrm{UC}(H)$, let $u \in \mathrm{UC}(H)$ be a viscosity subsolution (see [3] for the precise definition) of:

$$F(\nabla y) \leqq f(x) \qquad \text{in } H.$$

Then it is possible to show that $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ satisfy

$$F(\nabla v) \leqq f(x) + \mu(\varepsilon, \delta) \qquad \text{in } H$$

where $\mu(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0_+$.

(vi) We would like to mention that if $\varepsilon \geqq \varepsilon' > \delta' \geqq \delta > 0$ then one checks easily that

$$\underline{u}_{\varepsilon,\delta} \leqq \underline{u}_{\varepsilon',\delta'} \leqq u \leqq \bar{u}_{\varepsilon',\delta'} \leqq \bar{u}_{\varepsilon,\delta}.$$

Another inequality is obtained by remarking that we have

$$\begin{aligned} \underline{u}_{\varepsilon,\delta}(x) &\leqq \inf_{y \in H} \sup_{z \in H} \left[u(y) + \frac{1}{2\varepsilon} |y - z|^2 - \frac{1}{2\delta} |z - x|^2 \right] \\ &= \inf_{y \in H} \left[u(y) + \frac{1}{2(\varepsilon - \delta)} |y - x|^2 \right] \\ &= S_F(\varepsilon - \delta)u(x) \end{aligned}$$

while $\bar{u}_{\varepsilon,\delta} \geqq S_{-F}(\varepsilon - \delta)u$ on H.

(vii) Another property of the Inf-Sup convolutions $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ concerns critical points. Indeed, first of all, these regularizations preserve the symmetries of u: for instance, if u is even on H then $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ are also even. More generally, if u is invariant by a group of isometries of H, so are $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$. This fact is interesting in itself but also fundamental for critical point theory. Next, we remark that $S_{\pm F}(t)$ (for t small) preserves the critical points of u at least if $u \in C^{1,1}$.

Finally, it was observed in I. Ekeland and J. M. Lasry [5] that if u is semi-convex and satisfies (P.S.) condition then for t small $v = S_{-F}(t)u$ is $C^{1,1}$ and also satisfies (P.S.). Furthermore, ∇v may be used as a pseudo-gradient for u.

Applications to critical point theorems are given in [5] (see also A. Pommelet [8] for related considerations).

We conjecture that if u is Lipschitz (to simplify) and satisfies (P.S.) (with Clarke gradient), then $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ also satisfy (P.S.). This would enable one to employ critical point theory for nonsmooth functions via this regularization.

(viii) The last property of the inf-sup convolutions we wish to mention concerns the possibility of extending and regularizing a function u uniformly continuous on a subset K of H (this gives another proof of theorem 1 in [7]): indeed, consider

$$\underline{u}_{\varepsilon,\delta}(x)=\sup_{z\in H}\inf_{y\in K}\left[u(y)+\frac{1}{2\varepsilon}|y-z|^2-\frac{1}{2\delta}|z-x|^2\right],$$

then $\underline{u}_{\varepsilon,\delta}\in C^{1,1}(H)$, $u\geqq \underline{u}_{\varepsilon,\delta}\geqq u-m(t_\varepsilon)$ on K, $|\nabla_{\varepsilon,\delta}(x)|\leqq t_\varepsilon/\varepsilon$ on H.

Note also that if v is $C_b^{1,1}(H)$, then w defined by $w(x)=v(x)+k|x|^2$ is convex for large k, hence $v=w-k|\cdot|^2$ is the difference of two convex functions (compare with the approximation method in [7, theorem 1]).

II. Proofs

We first show the string of inequalities in (11): the first one is deduced from the inequality $u\geqq \inf_H u$, while the second one comes from the choice $y=x$ in the definition of $\underline{u}_{\varepsilon,\delta}$. The other inequalities are proved similarly.

Next, we observe that the explicit formula yields the fact that if u satisfies (10), then $S_{\pm F}(t)u$ also satisfies (10) for all $t\geqq 0$, thus proving (13).

We next remark that if u satisfies (10), then the infimum defining $S_F(\lambda)u(x)$ (resp. the supremum defining $S_{-F}(\lambda)u(x)$) for $\lambda>0$ may be restricted to points y satisfying

$$(16)\qquad |y-x|^2\leqq 2\lambda m(|y-x|),\quad\text{or}\quad |y-x|\leqq t_\lambda.$$

Indeed, consider for example $S_F(t)u(x)$; since $S_F(t)u\leqq u$ we may restrict the infimum to points y such that

$$u(y)+\frac{1}{2\lambda}|x-y|^2\leqq u(x)$$

and using (10) we deduce (16). And, since $S_F(\varepsilon)u\leqq \underline{u}_{\varepsilon,\delta}\leqq u$, (16) implies: $\underline{u}_{\varepsilon,\delta}\geqq u-m(t_\varepsilon)$, and (12) is proved. Notice also that (16) easily yields that if u satisfies (10), then $S_{\pm F}(\lambda)u$ is Lipschitz for $\lambda>0$ and

$$|S_{\pm F}(\lambda)u(x)-S_{\pm F}(\lambda)u(y)|\leqq\frac{t_\lambda}{\lambda}|x-y|,\qquad \forall x,y.$$

Recalling that $S_{\pm F}(t)$ preserves moduli of continuity for $t \geqq 0$, we deduce that $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta}$ are Lipschitz with $t_\varepsilon/\varepsilon$ as Lipschitz constant. This proves (14) (in a weak form at least).

It remains to show that $\underline{u}_{\varepsilon,\delta}$, $\bar{u}_{\varepsilon,\delta} \in C^{1,1}(H)$ and that (15) holds: we will prove these claims for $\underline{u}_{\varepsilon,\delta}$, the proof being identical for $\bar{u}_{\varepsilon,\delta}$. We first recall (from [6] for example) that if $u \in \mathrm{UC}(H)$, $S_F(t)u = v$ (resp. $S_{-F}(t)u$) is semi-concave (resp. semi-convex) and more precisely that we have

$$(17) \qquad v - \frac{1}{2t}|x|^2 \text{ is concave on } H \quad \left(\text{resp. } v + \frac{1}{2t}|x|^2 \text{ is convex on } H\right).$$

Indeed for each $y \in H$, the function

$$u(y) + \frac{1}{2t}|x-y|^2 - \frac{1}{2t}|x|^2$$

is affine in x and thus

$$v - \frac{1}{2t}|x|^2 = \inf_{y \in H}\left[u(y) + \frac{1}{2t}|x-y|^2 - \frac{1}{2t}|x|^2\right]$$

is concave on H. Hence $\underline{u}_\varepsilon$, $\underline{u}_{\varepsilon,\delta}$ satisfy

$$(17') \qquad \begin{aligned} &\underline{u}_\varepsilon - \frac{1}{2\varepsilon}|x|^2 \qquad \text{is concave on } H,\\ &\underline{u}_{\varepsilon,\delta} + \frac{1}{2\delta}|x|^2 \qquad \text{is convex on } H. \end{aligned}$$

We next want to show that $\underline{u}_{\varepsilon,\delta} - |x|^2/2(\varepsilon-\delta)$ is concave on H and this will again be a general property of $S_{-F}(t)$. Indeed, let $u \in \mathrm{UC}(H)$ satisfy

$$u - \frac{1}{2\lambda}|x|^2 \qquad \text{is concave on } H$$

for some $\lambda > 0$, then for $0 < t < \lambda$, $v = S_{-F}(t)u$ satisfies

$$v - \frac{1}{2(\lambda - t)}|x|^2 \qquad \text{is concave on } H.$$

This claim follows from the equality:

$$\begin{aligned} v(x) - \frac{1}{2(\lambda-t)}|x|^2 &= \sup_{y \in H}\left[u(y) - \frac{1}{2\lambda}|y|^2 + \frac{1}{2\lambda}|y|^2 - \frac{1}{2t}|x-y|^2 - \frac{1}{2(\lambda-t)}|x|^2\right]\\ &= \sup_{y\in H}\,[\Phi(x,y)] \end{aligned}$$

where $\Phi(x, y)$ is — as is easily checked — concave with respect to (x, y).

We conclude applying the elementary

LEMMA. *Let Φ be jointly concave in (x, y) on $H \times H$ and let $\psi(x) = \sup_{y \in H} \Phi(x, y) < \infty$, then ψ is concave on H.*

Indeed, let $x_1, x_2 \in H$, let $\varepsilon > 0$, choose y_1, y_2 in H such that

$$\psi(x_1) \leqq \Phi(x_1, y_1) + \varepsilon, \qquad \psi(x_2) \leqq \Phi(x_2, y_2) + \varepsilon;$$

then for $\theta \in [0, 1]$:

$$\begin{aligned}\psi(\theta x_1 + (1-\theta)x_2) &\geqq \Phi(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)\\ &\geqq \theta\Phi(x_1, y_1) + (1-\theta)\Phi(x_2, y_2)\\ &\geqq \theta\psi(x_1) + (1-\theta)\psi(x_2) - \varepsilon\end{aligned}$$

(the first inequality comes from the definition of ψ, the second from the joint concavity of Φ and the third from the choices of y_1, y_2). We conclude sending ε to 0.

In conclusion, we have proved that $\underline{u}_{\varepsilon,\delta}$ satisfies $\underline{u}_{\varepsilon,\delta} + \frac{1}{2}C_{\varepsilon,\delta}|x|^2$ is convex, $\underline{u}_{\varepsilon,\delta} - \frac{1}{2}C_{\varepsilon,\delta}|x|^2$ is concave. This yields that $\underline{u}_{\varepsilon,\delta} \in C^1(H)$ and we wish to show that this implies in fact $\underline{u}_{\varepsilon,\delta} \in C^{1,1}(H)$ and that (15) holds. This is well-known in finite dimensions but it seems to require a justification in general. Denote $v = \underline{u}_{\varepsilon,\delta}$, $C = C_{\varepsilon,\delta}$, let $x, y, \xi \in H$ and consider H_1 the vector space spanned by x, y, ξ. The restriction v_1 of v to H_1 still satisfies the semi-concavity and semi-convexity properties of v with the same constant C. Hence $v_1 \in C^{1,1}(H_1)$ and

$$|\nabla v_1(x) - \nabla v_1(y)| \leqq C|x - y|.$$

But $\nabla v_1(x) = P_1\nabla v(x)$, $\nabla v_1(y) = P_1\nabla v(y)$ where P_1 is the orthogonal projection onto H_1 and thus

$$|(\nabla v(x) - \nabla v(y), \xi)| \leqq C|x - y|\,|\xi|.$$

Since ξ is arbitrary, we conclude. ■

REFERENCES

1. V. Barbu and G. Da Prato, *Hamilton-Jacobi Equations in Hilbert Spaces*, Pitman, London, 1983.

2. M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Am. Math. Soc. **277** (1983), 1–42.

3. M. G. Crandall and P. L. Lions, in preparation.

4. M. G. Crandall, L. C. Evans and P. L. Lions, *Some properties of viscosity solutions of Hamilton–Jacobi equations*, Trans. Am. Math. Soc. **282** (1984), 487–502.

5. I. Ekeland and J. M. Lasry, *On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface*, Ann. Math. **112** (1980), 283–319.

6. P. L. Lions, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, London, 1982.

7. A. S. Nemirovski and S. M. Semenov, *The polynomial approximation of functions in Hilbert spaces*, Mat. Sb. (N.S.) **92** (134) (1973), 257–281.

8. A. Pommelet, *Transformée de Toland et la théorie de Morse pour certaines fonctions non différentiables*, Thèse de 3e cycle, Université Paris — Dauphine, 1984.